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ABSTRACT

Nonstandard Analysis gives an alternative approach to teaching elementary calculus. This paper hopes to communicate to the reader the ideas of this recent development in mathematics and its implications in teaching undergraduate students. The development of the approach is first briefly traced. Then a method of constructing on ordered field extension of the real number system is presented in detail. Recent comments on the approach are presented, and arguments for its use at the undergraduate level are discussed. (MNS)

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NONSTANDARD ANALYSIS

AND

ITS PLACE IN UNDERGRADUATE EDUCATION

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NONSTANDARD ANALYSIS

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Nonstandard Analysis deals with a nonstandard model R^*, an ordered field extension of the real number system R. R^* includes infinitely small and large numbers. Nonstandard Analysis is a relatively new subject developed by the logician Abraham Robinson in the early 1960's. He made rigorous the intuitively attractive infinitesimal calculus of Newton and Leibniz. Nonstandard Analysis gives an alternative approach of teaching elementary calculus. This paper hopes to communicate to the reader the ideas of this recent development in mathematics and its implications in teaching the students at the undergraduate level.

The origins of infinitesimals lie long before Leibniz published his first treatise on differential and integral calculus. In a memoire of six pages, in the Acta Eruditorum of 1684, nearly 300 years ago, Leibniz used infinitely small and large numbers in his arguments. Yet he was unable to explain clearly the nature of these mysterious quantities. Later on Cauchy and Weierstrass developed logically sound ϵ-δ methods of limits. Since the methods of limits are not quite intuitive, they took nearly 150 years to become widely accepted. However, Cauchy himself admitted that his ideas of limits were based on infinitesimals, eventhough he never mentioned them anywhere in his discussions.

Nonstandard Analysis

In 1960 Abraham Robinson developed an extension R^* of R using very sophisticated logic. In the last twenty five years slightly easier and more familiar methods of set theory have been developed.

Now we shall see one of the methods of constructing R^*. In a way similar to extending rationals to reals, one can construct the extended number system R^*, called hyperreals, by considering sequences of real numbers.

Let N be the set of natural numbers. Let f:N -> R be a real valued function. These are precisely the sequences of the form

$$(f(1), f(2), f(3), \ldots, f(i), \ldots).$$

We shall call the set of all such sequences as R^N. Defining pointwise addition and multiplication, R^N forms a ring. R^N contains a copy of R in the sequences of the form $(r, r, r, \ldots)$. $(1, 1, 1, \ldots)$ and $(0, 0, 0, \ldots)$ respectively are the multiplicative and the additive identities.
$f \geq g \iff f(i) \geq g(i)$ for all $i \in N$ defines a partial order. However, R^N has zero divisors. There are nonzero elements, such as, $(2, 0, 0, 0, \ldots)$, which have no multiplicative inverses. Thus R^N fails to be a field.

We redefine zeroes of R^N. We shall say $f \in R^N$ is zero if the subset S of N on which f is nonzero is 'small'. Let us define μ a probability measure on N as follows:

$$\mu(s) = \begin{cases} 1 & \text{if } S^c \text{ is finite;} \\ 0 & \text{if } S \text{ is finite.} \end{cases}$$

Thus $\mu(S) = 1 \rightarrow S$ is large and $\mu(S) = 0 \rightarrow S$ is small.
μ fails to define measures for infinite sets whose complements are also infinite. For example, if S = (even integers), S^c = (odd integers), we have to consider one of the sets to be large and the other to be small. Thus we

systematically classify each intermediate set as either small or large arbitrarily, subject only to constraints of consistency.

Let M = Set of all large sets = $\{A \mid A \subseteq N \text{ s.t. } \mu(A) = 1\}$

M satisfies the following properties:

(1) $\emptyset \notin M$ (proper filter).

(2) If $A \in M$ and $A \subseteq B$ then $B \in M$ (superset).

(3) If $A \in M$ and $B \in M$ then $A \cap B \in M$ (finite intersection).

(4) If $B \subseteq M$ then either $B \in M$ or $B^C \in M$ (maximality).

(5) M contains no finite subset of N (freeness).

M is called a free ultrafilter.

Now we define an equivalence relation on R^N.

For $A \in R^N$, $B \in R^N$ $\quad A \approx B \leftrightarrow \{n \mid A(n) = B(n)\} \in M$.

$\approx$ is an equivalence relation and $\approx$ creates a partition of R^N. The equivalence classes of R^N relative to $\approx$ are called hyperreal numbers and are denoted by R^*. R^* contains a copy of R. $r \in R$ corresponds to the equivalence class of $(r, r, r, \ldots)$. This subset of R^* which corresponds to R is called the set of standard elements. Elements of R^* which are not standard are called nonstandard elements, hence the name Nonstandard Analysis.

Define for $a, b, c \in R^*$,

$a + b = c \leftrightarrow \{n \mid A(n) + B(n)\} \in M$,

$a \cdot b = c \leftrightarrow \{n \mid A(n) \cdot B(n)\} \in M$,

$a \leq b \quad \leftrightarrow \{n \mid A(n) \leq B(n)\} \in M$,

where A, B, C are representatives of equivalence classes a, b and c respectively. These definitions are independent of the choices of representatives. R^* defined in this manner is a totally ordered field.

For, suppose $a \in R^*$ and $a \neq 0$. We show that a has an inverse in R^*. Let A be a representative of a in R^N. Then

$$B = \begin{cases} A^{-1}(n) \text{ if } A(n) \neq 0 \\ 1 \quad \text{if } A(n) = 0 \end{cases}$$ is the inverse of A.

If b is the equivalence class containing B then $a \cdot b = 1$.

Thus every nonzero element of R^* has an inverse in R^*.

Let $a \in R^*$. Define the absolute value of a by

$$|a| = \max(a, -a).$$

A number a is infinitely large <-> $|a| > n$ for all $n \in N$.

A number a is infinitely small <-> $|a| < n$ for all $n \in N$.

An infinitely small number is also called infinitesimal.

R^* contains infinitely large or infinitely small numbers.

For example, (1, 2, 3, ..., n,...) is an infinitely large number and

(1, 1/2, 1/3, ..., 1/n, ...) is an infinitesimal.

a, b $\in R^*$ are said to be infinitely close <-> a - b is an infinitesimal and we write $a \sim b$.

A number a is finite <-> $|a| < n$ for some $n \in N$.

The set of finite hyperreals is denoted by R^σ.

Every infinitesimal is finite, since it is less than 1. If $a \in R^*$ be a finite hyperreal and $\alpha = \sup_{x \in R, x \leq a} x$, then it can be proved that $a - \alpha$ is an infinitesimal. This implies that every finite hyperreal is infinitely close to a real number. In fact every $a \in R^\sigma$ is infinitely close to a unique $\alpha \in R$ given by $\alpha = \sup_{x \in R, x \leq a} x$ and write $\alpha = st(a)$. st stands for standard part of a hyperreal, which is a real number. st is in fact a real valued function,

$$st: R^\sigma \to R.$$

Observe the following facts:

0 is the only real (standard) infinitesimal.

Every infinitesimal is infinitely close to 0.

If f is a real valued function on R, we can define a function on R^*, denoted by f^*, in a natural way.

Let $a \in R^*$, $a = (A(1), A(2), \ldots)$, $A(i) \in R$. Define

$$f^*(a) = (f(A(1)), f(A(2)), \ldots).$$

We can also define hypernatural numbers N^*, and hyperrational numbers Q^*.

$$a \in N^* \leftrightarrow \{n \mid A(n) \text{ is a natural number}\} \in M.$$

$$a \in Q^* \leftrightarrow \{n \mid A(n) \text{ is a rational number}\} \in M.$$

R^* can be represented by a hyperreal line. Surrounding each real r, we introduce a collection of hyperreal numbers infinitely close to r. These numbers are all finite. Symbols such as ϵ, δ, Δx, Δy are used for nonzero infinitesimals. We can imagine that infinitely small portions of hyperreal line is seen through 'infintesimal microscopes' and infinite parts of hyperreal line is seen through 'infinite telescopes'.

We have now constructed a proper extension R^* of R. In the process we have also proved the existence of infinitely small and large numbers in R^*. Most of the properties of R are inherited by R^*. However, R^* is neither complete nor archimedean in the usual sense. The fact that the archimedean and completion properties do not hold in R^* is not a contradiction in itself. It only questions Leibniz's contention that what holds for the real numbers also holds for the extended numbers.

Recall that an ordered field F is called archimedean if for all $a \in F$, there exists $n \in N$, N being the set of natural numbers, such that $|a| < n$. The class $a \in R^*$ containing the sequence $(1, 2, 3, \ldots, i, \ldots)$, does not satisfy the archimedean property for any $n \in N$. R^* is nonarchimedean is not

surprising because a theorem of algebra states that, if a totally ordered field is archimedean then it is isomorphic to a subfield of R. Here R^* is much larger than R.

An ordered field F is said to be complete if for every subset $A \subseteq F$ bounded above has the least upper bound in F. Consider the set A of multiples of δ, where δ is an infinitesimal. $A = (m\delta \;/\; m \in N)$. A is bounded above by 1. Let b be an upper bound of A. But $b - k\delta$ is agin an upper bound of A. For, $(b - k\delta) - m\delta = b - (k + m)\delta \geq 0$. Thus, A has no least upper bound. Hence R^* is not complete in the usual sense.

The reason why completion axiom and archimedean property do not hold in R^* is because of the distinction made in logic between the first order and higher ordered sentences. In the first order sentences quantifiers are limited to real numbers. As far as first order sentences are concerned, you cannot distinguish between R and R^*. But some properties of real systems, such as completion and archimedean properties are higher order sentences. Robinson's approach precisely gives a rule as to the generalization of higher order sentences.

When Robinson says,"R^* has the same properties as R.", he refers to a specified collection of properties of R which can be formulated in a certain formal language of logic. The statements of this language have specific interpretation in R as well as R^*. The rule of reinterpretation, which is known as the *-transform, gives rise to a special class of sets called 'internal sets' which the formal language knows about. These internal sets have the same properties as R, while external sets do not. The general rule of interpretation is called the transfer principle.

Using the *-transform the archimedean principle which is ,"Given any

$x \in R$, there exists $n \in N$ such that $x < n$." is interpreted in R^* as, "Given any $x^* \in R^*$ there exists $n^* \in N^*$ such that $x^* < n^*$." Note that N^* contains infinitely large hypernatural numbers. Thus the archimedean property has a new interpretation in R^*.

It can be proved that the set of all infinitesimals is an external set. The proof is beyond the scope of this paper. The set of multiples of a single infinitesimal δ, $A = \{m\delta \mid \delta \text{ an infinitesimal}, m \in N\}$, which was discussed earlier, is a subset of the set of infinitesimals. Hence, R^* is not complete. However the completion property also has a new interpretation in R^*, which is as follows:

Suppose (A_n) is a sequence of elements of $P(R)$, where $P(R)$ is the set of subsets or the power set of R.
Define $f \in^* (A_n)$ iff $\{n \mid f(n) \in A_n\} \in M$, where M is the ultrafilter defined earlier. Define $(A_n)^* = \{f \mid f \in^* (A_n)\}$. $(A_n)^*$ is a subset of R^*. Such subsets are called internal subsets of R^*.

Suppose $(A_n)^*$ is nonempty and bounded above. Then there exists $f_0 \in R^*$ such that $f \leq f_0$ for all $f \in (A_n)^*$ and if $f \leq g$ for all $f \in (A_n)^*$, then $f_0 \leq g$. The proof is nontrivial.

Nonstandard Analysis has seemingly resolved the three hundred year old problem giving precise treatment of calculus using infinitesimals. Leibniz' contemporaries identified infinitesimals with zero, which led to contradictions that dy/dx is 0/0. In the present treatment a nonzero infinitesimal is infinitely close to zero and yet not equal to zero. In symbols $\epsilon \sim 0$ but $\neq 0$. When Leibniz used $dy/dx = f'(x)$, he meant dy and dx were infinitesimals themselves and he took dy/dx to be the ratio of infinitesimals. The formalism in Robinson's method is to define $f'(x) = dy/dx = st(\Delta y/\Delta x)$, which means

$\Delta y/\Delta x \sim dy/dx$. Now it is easier to define higher derivatives since $st(\Delta y/\Delta x)$ is once again a real valued function. These are some specific examples as to how Nonstandard Analysis cleared contradictions and notational problems of Leibniz to make infinitesimals a mathematically sound system.

In spite of occasinal inconsistencies, Leibniz' infinitesimal calculus was generally preferred because of its intuitive appeal. The method of limits was adopted, for it was the first mathematically precise treatment of calculus. Robinson's Nonstandard Analysis has also made the intuitive ideas of infinitesimals compatible with mathematical rigor. The ideas which originally motivated the discovery of calculus have been revived.

The concepts of limits took a long time to be widely accepted by teachers and students of calculus. Epsilon-delta methods are generally quite involved. On the other hand, with some explanations regarding the properties of R^*, the infinitesimal definitions of basic concepts of limits and continuity are simple and intuitive. For example, the definition of continuity of a function f at a point $x = c$ is simply that, if x is infinitely close to c, then f(x) is infinitely close to f(c). The definitions of the basic concepts became simpler and arguments more intuitive.

Based on Robinson's theory, Jerome Keisler has written a book, Elementary Calculus, using an axiomatic approach. Recently Henlé and Kleinberg have written another text titled Infinitesimal Calculus using elementary logic. These two books have made Robinson's infinitesimal methods available to college freshman. The attempts made by Keisler and Henlé and Kleinberg to introduce infinitely large and small numbers as quickly as possible for learning calculus are quite creditable.

Keisler's book was used in many experiments from 1969-1974. Sister Kathleen Sullivan's results are summarized in an article in the American Mathematical Monthly. Her conclusions are:

"Certainly there is no claim to have found an instant solution to the problem of teaching calculus. This is not 'calculus made easy.' Hard work and practice will still be required of the student, and a poor background in algebra will still make it difficult to learn calculus.

On the other hand there does seem to be considerable evidence to support the thesis that this is indeed a viable alternative approach to teaching calculus. (Any fears on the part of a would-be experimenter that students who learn calculus by way of infinitesimals will achieve less mastery of basic skills have been allayed.) It even appears highly probable that using the infinitesimal approach will make the calculus course a lot more fun both for teachers and students."

Errett Bishop, a staunch critic of the axiomatic approach, has written a review of Keisler's book in an issue of the Bulletin of the American Mathematical Society. He criticizes book by saying that it falls short on motivation and ezplanation of the axioms of hyperreal numbers. Criticism of Keisler's axiomatic method by Errett Bishop, who is a staunch constructivist, does not seem to be totally objective considering that Keisler has ventured in writing the first book ever using infinitesimals. Keisler seems to have questioned Paul Halmos's decision to ask a constructivist to review his book and compared it to asking a teetotaler to judge the taste of a new wine.

Dedekind and Weirstrass constructed real number system, extending the rational numbers. The word 'real' is used merely to distinguish the 'real' from the 'complex' numbers. The precise status of the real numbers is in fact quite obscure. Now that Robinson has extended the real number system to a

consistent model R^* to include infinitely large and infinitely small quantities, there is no reason, in principle, why we cannot use them. The ultimate decision however depends on their mathematical advantage and disadvantage.

The set of real numbers can be constructed from the set of rationals in many different ways, such as, Dedekind sections or Cauchy sequences. These different methods lead to isomorphic models of R because of the theorem: Any complete ordered field is isomorphic to R and Archimedean ordered field is isomorphic to some subfield of R containing Q. R also inherits all mathematical properties of Q in the same sense as they hold in Q; R has many more properties as compared to Q. The process of extending Q to R is not based on direct intuition from the natural world. Still real numbers have extremely convenient properties.

Just as R was extended from Q, R^* was extended from R. But the comparison ends there. R^* was first created in the context of sophisticated logic. The reason being, in the process of extension, R^* acquires many new properties, but not all properties of R are inherited in R^* in the same sense as it happens while extending Q to R. This fact is explained before. If we use a constructive approach, different methods lead to different models of R^*. This time we do not have any uniqueness theorem to remove the ambiguity as in the case of R. These different models of R^* may even have different cardinalities. First let us consider the disadvantages of using R^*. Since different methods do give rise to different infinitesimals and different extensions f^* of a real function f, questions such as "what is an infinitesimal?" or "what is f^*?" is difficult to answer, unless mathematicians would agree to use one particular model of R^*. Secondly, even though the existence of f^*, the extended function is guaranteed, one cannot always

express f^* algebraically or pictorially. The level of difficulty is comparable to that of axiom of choice. Thirdly, if a student does not have background in logic of the first order, they would not have a clear understanding of the nature of the infinitesimals.

I believe Robinson's theory provides foundations for modern anslysis and hence for calculus, which are as firm as the foundations that have evolved from Weierstrass. This does not imply a conflict either; we simply have two ways of looking at many things. Whatever our outlook is, it appears to me that the infinitesimals and infinitely large numbers of nonstandard model are neither more nor less legitimate than the irrational numbers, which we use so casually. As far as using this approach in teaching calculus, especially for the special student groups, such as Engineering, Physics and Mathematics majors, I am all for it and agree with Sister Kathleen Sullivan's conclusions, because they are based on actual experiments. Of course, more classroom experiments are needed to verify these results and to compare the new approach with the traditional approach. The most important thing is to know that this is no'calculus made easy', but it appears highly probable that using this approach might make the calculus course interesting for the teachers and for the students. Alternative to a purely axiomatic approach, one may use an approach based on a particular model of hyperreal numbers using some logic. However, switching from the new method to the traditional method is simply a waste of time and energy for students as well as teachers of calculus.

While Nonstandard Analysis could be introduced as a complete course at the undergraduate level, until teachers are trained to carry out this responsibility and proper books are written suitable for these students, the subject could be introduced gradually in a couple of chapters in any standard Analysis course. Students who are mature enough to understand or appreciate the

Dedekind cuts, axiom of choice and Cantor's method of Cauchy sequences may also be ready to understand concepts such as ultrafilters which one comes across in Nonstandard Analysis. It seems proper to teach the subject in detail at graduate level to prepare teachers qualified to teach Nonstandard Analysis and they in turn could introduce the subject fully at the elementary level.

Speaking about the general subject of nonstandard analysis a more positive opinion has been expressed by Kurt Gödel as quoted by Robinson in the preface of his book Nonstandard Analysis:

"...nonstandard analysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. It is a common misinterpretation of nonstandard analysis that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future."

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